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# The Simple Graphs Associated with Rings and Semigroups

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Let  $R$  be a commutative ring, and let  $Z(R)$  denote its set of zero-divisors. We associate a simple graph  $\Gamma(R)$  to  $R$  with vertices  $Z(R)^* = Z(R) - \{0\}$ , the set of nonzero zero-divisor of  $R$ . Two distinct vertices  $x$  and  $y$  are adjacent if  $xy = 0$ . This graphs are called the *zero-divisor graphs* of rings  $R$ .

We also associate a simple graph  $\Delta(\mathbf{Z}_n)$  to  $\mathbf{Z}_n$  with vertices  $\mathbf{Z}_n$  and for distinct elements  $x, y \in \mathbf{Z}_n$ , the vertices  $x$  and  $y$  are adjacent if and only if  $y = x^2$  ( $x \neq y$ ). This graphs are called the *parabola graphs*.

For a commutative multiplicative semigroup  $S$  with  $0$  ( $0x = 0$  for all  $x \in S$ ), we can defined the zero-divisor graph  $\Gamma(S)$  as above ([DMS]).

We denote an edge such that  $a$  and  $b$  are adjacent by  $a - b$ . We also denote a path by  $a - b - c - d$  etc. Also, let  $\chi(G)$  denote the chromatic number of the graph  $G$  and let  $\chi'(G)$  denote the edge chromatic number of the graph  $G$ .

The notion of a zero-divisor graph was first introduced by I. Beck in [B1] and further investigated in [A1], though their vertices set included the zero element. Let  $G$  be a graph.

The *diameter* of  $G$  is

$$\text{diam}(G) = \sup\{d(x, y) \mid x \text{ and } y \text{ are distinct vertices of } G\},$$

where  $d(x, y)$  denotes the length of the shortest path from  $x$  to  $y$ . The *grith* of  $G$ , denoted by  $g(G)$ , is defined as the length of the shortest cycle in  $G$ .

A complete subgraph of  $G$  is called *clique*.  $\omega(G)$ , the *clique number* of  $G$ , is the greatest integer  $r \geq 1$  such that  $K^r \subset G$ . Also,  $c(G)$ , the *circumference* of  $G$ , is the length of the longest cycle in  $G$ . Let  $n = p_1^{2n_1} \cdots p_k^{2n_k} q_1^{2m_1+1} \cdots q_r^{2m_r+1}$  for distinct primes  $p_i, q_j$  and integers  $n_i, m_j \geq 0$ . Then

$$\omega(\Gamma(\mathbf{Z}_n)) = p_1^{n_1} \cdots p_k^{n_k} q_1^{m_k} \cdots q_r^{m_r} + r - 1$$

<sup>1</sup> This is a part of an abstract and details will be published elsewhere.

by [B, Proposition 2.3].

For a graph  $G$  and an integer  $n \geq 1$ , we define  $\lambda(G, n)$  to be the number of complete subgraphs (cliques) of  $G$  of order  $n$ . Note that  $\lambda(G, 1)$  is the number of vertices of  $G$ ,  $\lambda(G, 2)$  is the number of edges of  $G$ ,  $\lambda(G, 3)$  is the number of triangles in  $G$  and  $\lambda(G, n) = 0$  for all  $n \geq \omega(G) + 1$  ([AFL]).

A graph  $G$  is *planar* if it can be drawn in such a way that no two edges intersect.

We examine the zero-divisor graph  $\Gamma(Z_n)$ , of the ring  $Z_n$  where the residue class ring modulo  $n$  where  $n$  is a positive integer. Many parts of this note are contained in [DS] and [AF].

We also investigate the parabola graphs.

For any commutative semigroup  $S$ , let  $C(S)$  = the *core* of  $\Gamma(S)$  be the union of the cycles in  $\Gamma(S)$ . A vertex  $x$  of  $\Gamma(S)$  is called an *end point* in case there is at most one edge in  $\Gamma(S)$  with vertex  $x$ .

## §1. Some examples of zero-divisor graphs and parabola graphs.

**Example 1.** Let  $S$  be the commutative nilsemigroup

$$S = \langle a, b \mid a^3 = a^2b = ab^2 = b^3 = 0 \rangle = \{a, b, a^2, ab, b^2, 0\}.$$

For this zero-divisor graph  $\Gamma(S) = (V(\Gamma(S)), E(\Gamma(S)))$ , we have that

$$V(\Gamma(S)) = \{a, b, a^2, ab, b^2\}$$

is the set of all vertices and  $E(\Gamma(S)) = \{a-a^2, a-b^2, a-ab, b^2-ab, b^2-a^2, b^2-b, \}$ . This graph is connected,  $d(a, b^2) = 1$ ,  $d(a, b) = 2$  and  $a - a^2 - b^2 - a$  is a cycle of length 3. And so  $\text{diam}(\Gamma(S)) = 2$  and  $g(\Gamma(S)) = 3$ . There is not an end point in  $\Gamma(S)$  and  $C(S) = \Gamma(S)$ . Also, we have that  $\omega(\Gamma(S)) = 4$ . Also we have that  $\lambda(S, 1) = 5$ ,  $\lambda(S, 2) = 9$  and  $\lambda(S, 2) = 6$ .

**Example 2.** Let  $S$  be the commutative nilsemigroup

$$S = \langle a, b \mid a^3 = a^2b = ab^2 = b^3 = 0, a^2 = b^2 \rangle = \{a, b, a^2, ab, 0\}.$$

$\Gamma(S)$  is connected,  $d(a, a^2) = 1$ ,  $d(a, b) = 2$  and  $a - a^2 - ab - a$  is a cycle of length 3. And so  $\text{diam}(\Gamma(S)) = 2$  and  $g(\Gamma(S)) = 3$ . There is not an end point in  $\Gamma(S)$  and  $C(S) = \Gamma(S)$ . Also, we have that  $\omega(\Gamma(S)) = 3$  and  $c(\Gamma(S)) = 3$ . Also we have that  $\lambda(S, 1) = 4$ ,  $\lambda(S, 2) = 5$  and  $\lambda(S, 2) = 2$ .

This graph  $\Gamma(S)$  is planar.

**Example 3.**  $V(\Gamma(\mathbf{Z}_{12}))$  of the zero-divisor graph  $\Gamma(\mathbf{Z}_{12})$  is the set

$$\{2, 3, 4, 6, 8, 9, 10\} \text{ and}$$

$$E(\Gamma(\mathbf{Z}_{12})) = \{2 - 6, 3 - 4, 3 - 8, 4 - 6, 4 - 9, 6 - 8, 6 - 10, 8 - 9\}.$$

The elements 4, 9 are idempotent elements and 6 is a nilpotent element. Also,  $\{0, 6\}$  is an ideal of  $\mathbf{Z}_{12}$ ,  $\omega(\Gamma(\mathbf{Z}_{12})) = 4$ . Also we have that  $\lambda(\mathbf{Z}_{12}, 1) = 7$ ,  $\lambda(\mathbf{Z}_{12}, 2) = 8$  and  $\lambda(\mathbf{Z}_{12}, 3) = 0$ . Also  $\text{diam}(\Gamma(\mathbf{Z}_{12})) = 3$ ,  $g(\Gamma(\mathbf{Z}_{12})) = 4$  and  $c(\Gamma(\mathbf{Z}_{12})) = 4$ . We have that  $\chi(\Gamma(\mathbf{Z}_{12})) = 2$  and  $\chi'(\Gamma(\mathbf{Z}_{12})) = 4$ .  $\Gamma(\mathbf{Z}_{12})$  is planar.

**Example 4.**  $V(\Gamma(\mathbf{Z}_{15}))$  of the zero-divisor graph  $\Gamma(\mathbf{Z}_{15})$  is the set

$$\{3, 5, 6, 9, 10, 12\} \text{ and}$$

$$E(\Gamma(\mathbf{Z}_{15})) = \{3 - 5, 3 - 10, 5 - 6, 5 - 9, 5 - 12, 6 - 10, 9 - 10, 10 - 12\}.$$

The elements 6, 10 are idempotent elements. The ring  $\mathbf{Z}_{15}$  has no ideals contained only two elements. We have that  $\omega(\mathbf{Z}_{15}) = 4$ ,  $c(\mathbf{Z}_{15}) = 4$  and also we have that  $\lambda(\mathbf{Z}_{15}, 1) = 6$ ,  $\lambda(\mathbf{Z}_{15}, 2) = 8$  and  $\lambda(\mathbf{Z}_{15}, 2) = 0$ . Also  $\text{diam}(\Gamma(\mathbf{Z}_{15})) = 2$  and  $g(\Gamma(\mathbf{Z}_{15})) = 4$ . We have that  $\chi(\Gamma(\mathbf{Z}_{15})) = 2$  and  $\chi'(\Gamma(\mathbf{Z}_{15})) = 4$ .  $\Gamma(\mathbf{Z}_{15})$  is planar.

**Example 5.**  $V(\Gamma(\mathbf{Z}_{16}))$  of the zero-divisor graph  $\Gamma(\mathbf{Z}_{16})$  is the set

$$\{2, 4, 6, 8, 10, 12, 14\} \text{ and}$$

$$E(\Gamma(\mathbf{Z}_{16})) = \{2 - 8, 4 - 8, 4 - 12, 6 - 8, 8 - 10, 8 - 12, 8 - 14\}.$$

The element 8 is an idempotent element and 4, 12 are nilpotent elements. The set  $\{0, 8\}$  is an ideal of  $\mathbf{Z}_{16}$ . We have that  $\omega(\mathbf{Z}_{16}) = 3$ . Also we have that  $\lambda(\mathbf{Z}_{16}, 1) = 7$ ,  $\lambda(\mathbf{Z}_{16}, 2) = 7$  and  $\lambda(\mathbf{Z}_{16}, 3) = 1$ . Also  $\text{diam}(\Gamma(\mathbf{Z}_{16})) =$

$2, g(\Gamma(\mathbf{Z}_{16})) = 3$  and  $c(\Gamma(\mathbf{Z}_{16})) = 3$ . We have that  $\chi(\Gamma(\mathbf{Z}_{16})) = 3$  and  $\chi'(\Gamma(\mathbf{Z}_{16})) = 6$ .  $\Gamma(\mathbf{Z}_{16})$  is planar.

**Example 6.** Let  $\Gamma(\mathbf{Z}_4[X]/(X^2)) = \Gamma(\mathbf{Z}_4[x])$  ( $x^2 = 0$ ) be a zero-divisor graph associated with the ring  $\mathbf{Z}_4[X]/(X^2)$ . Set  $R_4[x] = \mathbf{Z}_4[x]$ . Then  $V(\Gamma(R_4[x])) = \{2, x, 2+x, 2x, 2+2x, 3x, 2+3x\}$  and  $E(R_4[x]) = \{2-2x, 2-2+2x, x-2x, x-3x, 2+x-2x, 2x-2+2x, 2+3x-2x\}$ .

We have that  $\omega(R_4[x]) = 3, c(R_4[x]) = 3$  and also we have that  $\lambda(R_4[x], 1) = 7, \lambda(R_4[x], 2) = 8, \lambda(R_4[x], 3) = 2$  and  $\lambda(R_4[x], 4) = 0$ . Also  $\text{diam}(\Gamma(R_4[x])) = 2$  and  $g(\Gamma(R_4[x])) = 3$ . We have that  $\chi(\Gamma(R_4[x])) = 3$  and  $\chi'(\Gamma(R_4[x])) = 6$ .

$\Gamma(R_4[x])$  is planar.

**Lemma 1.**  $\Gamma(\mathbf{Z}_n)$  is connected and  $\text{diam}(\mathbf{Z}_n) \leq 3$ .

**Lemma 2.** If  $\Gamma(\mathbf{Z}_n)$  contains a cycle, then  $g(\mathbf{Z}_n) \leq 4$ .

**Conjecture.** The number of end points of  $\Gamma(\mathbf{Z}_n)$  is an even number.

This conjecture is valid for  $n \leq 30$ .

**Lemma 3.** If  $n \geq 9$  and  $\Gamma(\mathbf{Z}_n)$  is a zero-divisor graph of  $\mathbf{Z}_n$ , then there exists an element  $x$  such that  $\{0, x\}$  is an ideal of  $\mathbf{Z}_n$ ,  $\text{Ann}(x)$  is a maximal ideal of  $\mathbf{Z}_n$  and  $\mathbf{Z}_n/\text{Ann}(x) \cong \mathbf{Z}_2$ .

If  $p$  is a prime number ( $\neq 2$ ), then  $\{0, p\}$  is an ideal of  $\mathbf{Z}_{2p}$ . So  $\Gamma(\mathbf{Z}_{2p})$  is a star graph.

**Lemma 4.** If any vertex in  $\Gamma(\mathbf{Z}_n)$  is either a vertex of the core  $C(\Gamma(\mathbf{Z}_n))$  or else is an end point of  $\Gamma(\mathbf{Z}_n)$ .

**Lemma 5** ([B] and [AN]). Let  $p, q$  and  $r$  be all distinct prime numbers. The following statements are satisfied.

- (1)  $\omega(\Gamma(\mathbf{Z}_n)) = 1$  if and only if  $n = 4$ .
- (2)  $\omega(\Gamma(\mathbf{Z}_n)) = 2$  if and only if  $n = 8, 9, pq, 4p$  ( $p \neq 2$ ).
- (3)  $\omega(\Gamma(\mathbf{Z}_n)) = 3$  if and only if  $n = prq, 4pq$  ( $p \neq 2, q \neq 2$ ),  $8p$  ( $p \neq 2$ ),  $9p$  ( $p \neq 3$ ),  $16, 27$ .

## §2. The parabola graph $y = x^2$

We associate a simple graph  $\Delta(\mathbf{Z}_n)$  to  $\mathbf{Z}_n$  with vertices  $\mathbf{Z}_n$  and for distinct elements  $x, y \in \mathbf{Z}_n$ , the vertices  $x$  and  $y$  are adjacent if and only if  $y = x^2$  ( $x \neq y$ ).

**Example 1.** The vertices set  $V(\Delta(\mathbf{Z}_7))$  of the simple graph  $\Delta(\mathbf{Z}_7)$  is the set  $\mathbf{Z}_7$  and  $E(\Delta(\mathbf{Z}_7)) = \{1 - 6, 2 - 3, 2 - 4, 4 - 5\}$ .  $\lambda(\Delta(\mathbf{Z}_7), 1) = 7, \lambda(\Delta(\mathbf{Z}_7), 2) = 4$  and  $\lambda(\Delta(\mathbf{Z}_7), 3) = 0$ . We have that  $\text{diam}(\Delta(\mathbf{Z}_7)) = 3, \omega(\Delta(\mathbf{Z}_7)) = 2, c(\Delta(\mathbf{Z}_7)) = g(\Delta(\mathbf{Z}_7)) = 0$ .

**Example 2.** Let  $\Delta(\mathbf{Z}_{11}) = (V(\Delta(\mathbf{Z}_{11})), E(\Delta(\mathbf{Z}_{11})))$  be a simple graph associated with  $\mathbf{Z}_{11}$ . We have that  $V(\Delta(\mathbf{Z}_{11})) = \mathbf{Z}_{11}$  and  $E(\Delta(\mathbf{Z}_{11})) = \{2 - 10, 3 - 5, 3 - 6, 3 - 9, 4 - 5, 4 - 9, 5 - 7, 8 - 9\}$ .  $\lambda(\Delta(\mathbf{Z}_{11}), 1) = 11, \lambda(\Delta(\mathbf{Z}_{11}), 2) = 8$  and  $\lambda(\Delta(\mathbf{Z}_{11}), 3) = 0, \lambda(\Delta(\mathbf{Z}_{11}), 4) = 1$ . We have that  $\text{diam}(\Delta(\mathbf{Z}_{11})) = 4, \omega(\Delta(\mathbf{Z}_{11})) = 2, c(\Delta(\mathbf{Z}_{11})) = 4, g(\Delta(\mathbf{Z}_{11})) = 4$ . This graph is not a forest.

**Theorem.** (a) Let  $\Delta(\mathbf{Z}_7)$  be a parabola graph and let  $A_7$  be an adjacent matrix of a parabola graph  $\Delta(\mathbf{Z}_7)$ . Also,  $f_n(X) = f_{A^n}(X)$  be a minimal polynomial of  $A_7$ . Then the following statements hold.

- (1) If  $n$  is an even natural number, then  $f_n(X)$  has a divisor  $X^2 - L_n X + 1$ .
- (2) If  $n$  is an odd natural number, then  $f_n(X)$  has a divisor  $X^2 - L_n X - 1$ , where  $L_n$  is a Lucas number.

(b) (1)  $\Delta(\mathbf{Z}_p)$  has no triangles, that is,  $\lambda(\Delta(\mathbf{Z}_p), 3) = 0$  for a prime number  $p$ .

(2)  $\lambda(\Delta(\mathbf{Z}_n)) = 0$  for  $1 \leq n \leq 10$  and  $12 \leq n \leq 20$ .

(3) For parabola graphs  $\Delta(\mathbf{Z}_n)$  ( $2 \leq n \leq 20$ ), their graphs are forest except  $n = 11$ .

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